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BENDING OF A PERFORATED CIRCULAR CYLINDRICAL CANTILEVER

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Abstract—Employing a special class of basis functions, a Saint-Venant flexure function suitable for the problem of the bending of a circular cylinder with $4N$ ($N = 1, 2, 3, \ldots$) circular cylindrical cavities in the axial direction is obtained. It is assumed that the load passes through the centroid of the symmetrical cross-section of the beam. It is also assumed that all conditions needed for the Saint-Venant solution of the bending of prismatic bars with one end fixed are met. The linearly-independent basis functions which automatically satisfy a homogeneous outer boundary condition are derived in closed form. They are generated from the integration of the products of trigonometric functions and the Green's function for the solution of Poisson's equation with singularities symmetrically located in four quadrants. Through shrinking the radius of the closed path of integration and by certain limiting processes, the integrations are performed analytically. The functions so obtained are summed to closed forms by the employment of an analytic function and its derivatives. The mathematical proof for the continuity of the functions and their derivatives across the circle joining the centers of the holes is presented. The inner boundary condition(s) is (are) satisfied with the point-by-point technique and method of least square error. Numerical results for the case of a beam with four circular cylindrical cavities are given.

INTRODUCTION

Analytical solutions for the flexure of circular cylindrical beams with one eccentric circular cylindrical cavity, according to the Saint-Venant theory, have been obtained in a few investigations. It was first suggested by Love (1906) that the classical Saint-Venant flexure functions may be found as series expansions in suitable curvilinear coordinates. Following this suggestion, Young *et al.* (1918) wrote down the form of the solution in series for the case in which the load is at right angles to the axis of symmetry of the cross-section. Later on, Seth (1936) gave the solution for the flexure functions for the cases where the load is resolved along and perpendicular to the axis of symmetry. Stevenson (1949) solved the general flexure problem for a hollow beam bounded by two eccentric circles. However, it is believed that the analytical solution of the bending of a circular cylinder with multiple cylindrical cut-outs has not yet been presented in the literature. In this investigation the solution to the title problem is obtained in the following manner. First, the Green's function for four symmetrically-located point sources in the circular region is derived. This Green's function automatically satisfies a homogeneous outer boundary condition. The result is now multiplied by appropriate trigonometric functions, and the products are integrated over a small circular path. Shrinking the radius of the path of integration and employing certain limiting processes, the integrations are analytically performed leading to linearly-independent basis functions in the form of infinite series. The series solutions for the basis functions are then summed to closed form using an analytic function and its derivatives. For the cases of eight or 12, or more cylindrical cavities one can repeat this process with the symmetrically-located point sources being placed inside the new cut-outs. Next, the aforementioned basis functions are multiplied by unknown constants and are added to the well-known solution of the problem of the bending of a solid circular cross-section beam given by Timoshenko and Goodier (1951) and Sokolnikoff (1956). Finally, the unknown constants involved are determined by satisfying the inner boundary condition(s) numerically. The technique employed here was briefly discussed by Naghdi (1988). However, in this investigation the basis functions are reduced to much simpler forms. The simplification has enabled the author to write a computer program which calculates the values of closed-form basis functions of any arbitrary order. With this program the problems of the bending of a cylinder with eight or 12 cut-outs, etc.,. . . . can easily be obtained. It is proved that the basis functions and their derivatives are continuous across the circular line joining the centers of the cut-outs. Numerical values of dimensionless shear stresses in the direction of the load for the case of a beam with four cavities are presented.

METHOD OF SOLUTION

In the following, we shall discuss the fundamental sets of basis functions and the solution for the case of a beam with four cylindrical cut-outs ($N = 1$). The basis functions for $N > 1$ are obtained by placing the poles of the fundamental sets at the centers of the new cavities and thus generating the appropriate solution.

Consider a prismatic cantilever beam whose cross-section is a multiply-connected circular region with radius R having four symmetrically-located circular cut-outs. Choose the centroidal nondimensional coordinate axes $\xi = X/R$, $\eta = Y/R$ as well as the special polar coordinates $\rho = r/R$ and θ as shown in Fig. 1 such that

$$\begin{aligned} \xi &= \rho \sin \theta, \\ \eta &= -\rho \cos \theta, \end{aligned} \qquad 0 \leqslant \rho \leqslant 1. \tag{1}$$

It is assumed that the line of action of the load W coincides with the ξ axis.

According to the Saint-Venant theory of the bending of prismatic bars given in Sokolnikoff (1956), the equation

$$\nabla^2 \Phi = 0 \tag{2}$$

must be satisfied and the boundary conditions

$$\frac{d\Phi}{dn_i} = -[\tfrac{1}{2}\nu\xi^2 + (1 - \tfrac{1}{2}\nu)\eta^2] \cos(\xi, n_i) - (2+\nu)\xi\eta \cos(\eta, n_i), \quad i = 0, 1, 2, 3, 4 \tag{3}$$

have to be fulfilled. Here in relation (3) ν is Poisson's ratio, Φ is Saint-Venant's flexure

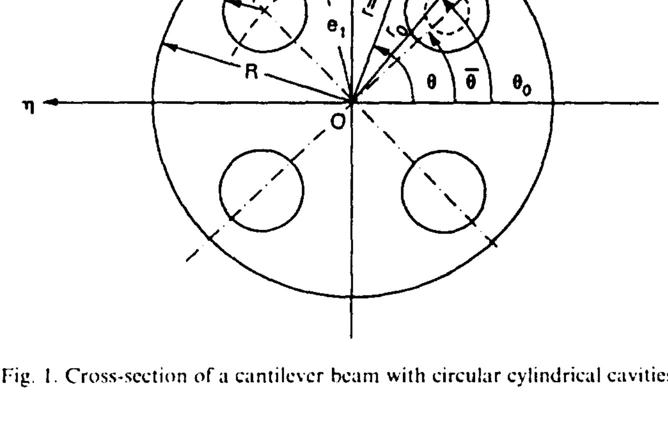

Fig. 1. Cross-section of a cantilever beam with circular cylindrical cavities.

function in dimensionless form, and n_i are the directions of the outward normal to the outer and inner boundaries. The function Φ is now sought in the form

$$\Phi = \tilde{\Phi}+A_0\Phi_0^*+A_1\Phi_1^*+A_2\Phi_2^*+\cdots+B_1\Phi_1^0+B_2\Phi_2^0+\cdots, \tag{4}$$

in which $\tilde{\Phi}$ is the flexure function for the bending of a solid circular cross-section beam given by Sokolnikoff (1956):

$$\tilde{\Phi} = -\left(\frac{3}{4}+\frac{\nu}{2}\right)\xi+\frac{1}{4}(\xi^3-3\xi\eta^2), \tag{5}$$

and Φ_J^* and Φ_J^0 are sets of closed-form linearly-independent basis functions of Laplace's equation which automatically satisfy the conditions

$$\left.\begin{aligned}\frac{\partial\Phi_J^*}{\partial\rho} &= 0, \quad J = 0,1,2,\ldots,\\ \frac{\partial\Phi_J^0}{\partial\rho} &= 0, \quad J = 1,2,\ldots,\end{aligned}\right\} \text{at} \quad \rho = 1 \tag{6}$$

and have singularities at the centers of the cut-outs. The unknown constants $A_0, A_1, \ldots, B_1, B_2$, are to be determined from the inner boundary condition.

For the derivation of Φ_J^* and Φ_J^0 the following technique is employed. First, the solution of Poisson's equation suitable for the problem under consideration:

$$\nabla^2\tilde{\Phi} = \sum_{n=1,3,5}^{\infty} S_n^*\delta(\rho-\rho_0)\sin n\theta, \tag{7}$$

in which S_n^* are the Fourier coefficients and δ is the unit impulse function, is sought in the form

$$\tilde{\Phi} = \sum_{n=1,3,5}^{\infty} f_n(\rho)\sin n\theta. \tag{8}$$

Substituting (8) in (7), employing the condition

$$\frac{\mathrm{d}f_n(\rho)}{\mathrm{d}\rho} = 0 \quad \text{at outer boundary } \rho = 1, \tag{9}$$

and considering that $f_n(\rho)$ must be continuous at $\rho = \rho_0$, we find the function $\tilde{\Phi}$ with positive sources at ($\rho = \rho_0, \theta = \theta_0$ and $\theta = \pi-\theta_0$) and negative sources at ($\rho = \rho_0, \theta = -\theta_0$ and $\theta = \pi+\theta_0$):

$$\begin{aligned}\tilde{\Phi} &= \sum_{n=1,3,5}^{\infty} -\frac{2S}{R\pi n}[\rho^n\rho_0^n+\rho^{-n}\rho_0^n]\sin n\theta_0\sin n\theta, \quad \text{for} \quad \rho > \rho_0\\ \tilde{\Phi} &= \sum_{n=1,3,5}^{\infty} -\frac{2S}{R\pi n}[\rho^n\rho_0^{-n}+\rho^n\rho_0^n]\sin n\theta_0\sin n\theta, \quad \text{for} \quad \rho < \rho_0,\end{aligned} \tag{10}$$

in which S is the magnitude of the concentrated sources.

Next, we shall generate certain basis functions Φ_J^* and Φ_J^0 from the following integrals:

$$\Phi_J^* = \oint \bar{\Phi}(\rho, \theta, \rho_0, \theta_0)\varepsilon \cos J\beta_0 \, d\beta_0, \quad J = 0, 1, 2, \ldots, \tag{11}$$

$$\Phi_J^0 = \oint \bar{\Phi}(\rho, \theta, \rho_0, \theta_0)\varepsilon \sin J\beta_0 \, d\beta_0, \quad J = 1, 2, \ldots, \tag{12}$$

in which the path of integration is around the center of one of the four inner circular cavities with parametric equations (see Fig. 1)

$$\begin{aligned} &\rho_0 = a_0(1+\varepsilon\cos\beta_0), \quad \theta_0^* = \varepsilon\sin\beta_0, \quad a_0 = e_1/R, \\ &\theta_0^* = \theta_0 - \theta, \quad \varepsilon = \text{a small positive number.} \end{aligned} \tag{13}$$

The analytical evaluations of the integrals in relations (11) and (12) are quite involved for a finite value of ε. However, it is possible to obtain these integrals when ε tends to zero. Note that ρ and θ are considered constants in these integrals and the terms involved are series like

$$\sum_{n=1,3,5}^{\infty} \frac{(\rho\rho_0)^n}{n} \sin n\theta \sin n\theta_0. \tag{14}$$

We shall only show the integration involving series (14):

$$\Gamma_J = \sum_{n=1,3,5}^{\infty} \left[\rho^n \sin n\theta \int_0^{2\pi} \frac{\rho_0^n}{n} \sin n\theta_0 \varepsilon \sin J\beta_0 \, d\beta_0\right]. \tag{15}$$

Substituting now eqn (13) into relation (15) we obtain

$$\begin{aligned} \Gamma_J = &\sum_{n=1,3,5}^{\infty} \left[a_0^n \rho^n \sin n\theta \cos n\theta \int_0^{2\pi} \frac{(1+\varepsilon\cos\beta_0)^n}{n} \sin(n\varepsilon\sin\beta_0)\varepsilon\sin J\beta_0 \, d\beta_0\right] \\ &+ \sum_{n=1,3,5}^{\infty} \left[a_0^n \rho^n \sin n\theta \sin n\theta \int_0^{2\pi} \frac{(1+\varepsilon\cos\beta_0)^n}{n} \cos(n\varepsilon\sin\beta_0)\varepsilon\sin J\beta_0 \, d\beta_0\right]. \end{aligned} \tag{16}$$

Expanding $\sin(n\varepsilon\sin\beta_0)$, $\cos(n\varepsilon\sin\beta_0)$ in the Fourier series given by Dwight (1957), $(1+\varepsilon\cos\beta_0)^n$ in powers of $\varepsilon\cos\beta_0$, and considering that the second integral in relation (16) is zero, we get:

$$\begin{aligned} \Gamma_J = \sum_{n=1,3,5}^{\infty} \frac{a_0^n}{n} \rho^n \sin n\theta \cos n\theta \int_0^{2\pi} &[\alpha_0 + \alpha_1\varepsilon\cos\beta_0 + \alpha_2\varepsilon^2\cos^2\beta_0 + \cdots] \\ &\cdot [J_0(n\varepsilon) + 2J_2(n\varepsilon)\cos 2\beta_0 + \cdots]\varepsilon\sin j\beta_0 \, d\beta_0, \end{aligned} \tag{17}$$

in which $J_0(n\varepsilon)$, $J_2(n\varepsilon), \ldots$ are the Bessel functions of the first kind, and $\alpha_0, \alpha_1, \alpha_2, \ldots$ are obtained from the binomial expansion of $(1+\varepsilon\cos\beta_0)^n$:

$$\begin{aligned} &\alpha_0 = 1, \quad \alpha_1 = n, \quad \alpha_2 = \frac{1}{2!}(n^2-n), \quad \alpha_3 = \frac{1}{3!}(n^3-3n^2+2n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{18}$$

Note that the terms which are the results of the product of the two brackets in relation (17) will produce a nonzero integral only if they contain the $\sin J\beta_0$ term. As ε tends to zero, and the smaller terms of higher order are dropped, analytical integration leads to

$$\Gamma_1 = \frac{\pi}{2}\varepsilon \sum_{n=1,3,5}^{\infty} a_0^n \rho^n [\sin n\beta_1 + \sin n\beta_2],$$

$$\Gamma_2 = \left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right)\varepsilon^2 \sum_{n=1,3,5}^{\infty} n a_0^n \rho^n [\sin n\beta_1 + \sin n\beta_2],$$

$$\Gamma_3 = \frac{\pi}{2}\left(\frac{1}{3!}\right)\frac{1}{(2)^2}\varepsilon^3 \sum_{n=1,3,5}^{\infty} (4n^2 - 3n) a_0^n \rho^n [\sin n\beta_1 + \sin n\beta_2],$$

$$\cdots\cdots\cdots,$$

$$\beta_1 = \theta + \theta_0, \quad \beta_2 = \theta - \theta_0. \tag{19}$$

Evaluating other integrals in the same fashion, and combining the results, we obtain a set of linearly-independent basis functions as given in the following:

$$\Phi_0^* = \sum_{n=1,3,5}^{\infty} \frac{1}{n}[(a_0\rho)^n + (a_0/\rho)^n][\cos n\beta_1 - \cos n\beta_2],$$

$$\Phi_1^* = \sum_{n=1,3,5}^{\infty} \frac{1}{n}[n(\rho a_0)^n + n(a_0/\rho)^n][\cos n\beta_1 - \cos n\beta_2],$$

$$\Phi_2^* = \sum_{n=1,3,5}^{\infty} \frac{1}{n}[(2n^2 - n)(\rho a_0)^n + (2n^2 - n)(a_0/\rho)^n][\cos n\beta_1 - \cos n\beta_2],$$

$$\Phi_3^* = \sum_{n=1,3,5}^{\infty} \frac{1}{n}[(4n^3 - 3n^2 + 2n)(\rho a_0)^n + (4n^3 - 3n^2 + 2n)(a_0/\rho)^n][\cos n\beta_1 - \cos n\beta_2],$$

$$\cdots\cdots\cdots \quad \text{for} \quad \rho > a_0, \quad J = 0,1,2,\ldots. \tag{20}$$

Similarly, the functions $\Phi_0^*, \Phi_1^*, \ldots$ for $\rho < a_0$, as well as the functions Φ_J^0 for both regions $\rho > a_0$ and $\rho < a_0$, may be easily written. However, for the sake of brevity they shall not be reproduced here.

Note that for simplicity, certain constant coefficients such as π, S, R, ε^J have been dropped from the basis functions.

Finally, it is seen from relation (20) that the basis functions can be written in much simpler forms:

$$\Phi_J^* = \sum_{n=1,3,5}^{\infty} \frac{1}{n}[n^J(\rho a_0)^n + n^J(a_0/\rho)^n][\cos n\beta_1 - \cos n\beta_2],$$

$$\text{for} \quad \rho > a_0, \quad J = 0, 1, 2, \ldots \tag{21}$$

$$\Phi_J^* = \sum_{n=1,3,5}^{\infty} \frac{1}{n}[n^J(\rho a_0)^n + (-1)^J n^J(\rho/a_0)^n][\cos n\beta_1 - \cos n\beta_2],$$

$$\text{for} \quad \rho < a_0, \quad J = 0, 1, 2, \ldots \tag{22}$$

$$\Phi_J^0 = \sum_{n=1,3,5}^{\infty} \frac{1}{n}[n^J(\rho a_0)^n + n^J(a_0/\rho)^n][\sin n\beta_1 + \sin n\beta_2],$$

$$\text{for} \quad \rho > a_0, \quad J = 1, 2, 3, \ldots \tag{23}$$

$$\Phi_J^0 = \sum_{n=1,3,5}^{\infty} \frac{1}{n}[n^J(\rho a_0)^n + (-1)^{J+1} n^J(\rho/a_0)^n][\sin n\beta_1 + \sin n\beta_2],$$

$$\text{for} \quad \rho < a_0, \quad J = 1, 2, 3, \ldots. \tag{24}$$

It is seen from eqns (21) through (24) that the first partial derivatives with respect to ρ of the basis functions Φ_J^*, Φ_J^0 contain series such as

$$\sum_{n=1,3,5}^{\infty} n^J(\rho a_0)^n \begin{pmatrix}\cos n\beta \\ \sin n\beta\end{pmatrix}, \quad \sum_{n=1,3,5}^{\infty} n^J(a_0/\rho)^n \begin{pmatrix}\cos n\beta \\ \sin n\beta\end{pmatrix}, \quad \sum_{n=1,3,5}^{\infty} n^J(\rho/a_0)^n \begin{pmatrix}\cos n\beta \\ \sin n\beta\end{pmatrix}. \tag{25}$$

These series in turn can be rewritten in the form:

$$\sum_{n=1,3,5}^{\infty} n^J \mathrm{e}^{-n\lambda} \begin{pmatrix}\cos n\beta \\ \sin n\beta\end{pmatrix}, \tag{26}$$

in which λ is a positive number which takes values:

$$\begin{aligned}\lambda &= -\ln(\rho a_0),\\ \lambda &= -\ln(a_0/\rho),\\ \lambda &= -\ln(\rho/a_0).\end{aligned} \tag{27}$$

We shall show in the following that the series given in eqn (26) can be summed in closed forms. Consider the analytic function (see Gradshteyn and Ryzhik, 1965)

$$F(z) = \tfrac{1}{2}+\sum_{n=1}^{\infty} e^{-nz} = \tfrac{1}{2}\coth(z/2), \quad z = \lambda+\mathrm{i}\beta, \quad \lambda > 0, \quad \mathrm{i} = \sqrt{-1}. \tag{28}$$

Since the derivatives of an analytic function are still analytic functions, we differentiate (28) to obtain

$$\begin{aligned}\sum_{n=1}^{\infty} n\,\mathrm{e}^{-nz} &= \frac{1}{4}\frac{1}{\sinh^2(z/2)},\\ \sum_{n=1}^{\infty} n^2\mathrm{e}^{-nz} &= \frac{1}{4}\frac{\cosh(z/2)}{\sinh^3(z/2)},\\ \sum_{n=1}^{\infty} n^3\mathrm{e}^{-nz} &= \frac{[3\coth^2(z/2)-1]}{8\sinh^2(z/2)},\\ &\ldots\ldots\ldots\ldots\end{aligned} \tag{29}$$

The real and imaginary parts of the functions involved in eqn (29) produce the closed-form representations of the series given in relation (26).

We have developed a recurrence formula for the nth derivative of the function $F(z)$ and programmed it. Thus, the closed-form values of the basis functions Φ_J^* and Φ_J^0 are available for any given J.

PROOF OF THE CONTINUITY OF THE FUNCTIONS

The apparent forms of Φ_J^* or Φ_J^0 for $\rho > a_0$ and $\rho < a_0$ raises the question of the continuity of these functions and their partial derivatives at $\rho = \rho_0$. We shall show in the following that actually, the two forms of basis functions for $\rho > \rho_0$ and $\rho < \rho_0$ are mathematically identical when ρ tends to ρ_0, and therefore the basis functions and their partial derivatives of any order are continuous at $\rho = \rho_0$. Noting relations (11) and (12), it shall be sufficient to show this continuity for the Green's function $\tilde{\Phi}$. Rewriting series (10), we have

$$\tilde{\Phi} = \frac{S}{\pi R}\sum_{n=1,3,5}^{\infty}\frac{1}{n}[e^{-n\lambda_1}+e^{-n\lambda_2}][\cos n\beta_1 - \cos n\beta_2]$$
$$= \frac{S}{2\pi R}\sum_{n=1,2,3}^{\infty}\left\{\frac{1}{n}[e^{-n\lambda_1}+e^{-n\lambda_2}][\cos n\beta_1 - \cos n(\beta_1+\pi) - \cos n\beta_2 + \cos n(\beta_2+\pi)]\right\}$$
$$\text{for} \quad \rho > \rho_0, \tag{30}$$

$$\lambda_1 = -\ln(\rho\rho_0), \quad \lambda_2 = -\ln(\rho_0/\rho),$$

$$\tilde{\Phi} = \frac{S}{\pi R}\sum_{n=1,3,5}^{\infty}\frac{1}{n}[e^{-n\lambda_1}+e^{-n\lambda_3}][\cos n\beta_1 - \cos n\beta_2]$$
$$= \frac{S}{2\pi R}\sum_{n=1,2,3}^{\infty}\{[e^{-n\lambda_1}+e^{-n\lambda_3}][\cos n\beta_1 - \cos n(\beta_1+\pi) - \cos n\beta_2 + \cos n(\beta_2+\pi)]\}$$
$$\text{for} \quad \rho < \rho_0, \tag{31}$$

$$\lambda_3 = -\ln(\rho/\rho_0).$$

It was shown by Naghdi (1973) that

$$\sum_{n=1,2,3}^{\infty}\frac{1}{n}e^{-n\lambda}\cos n\beta = \tfrac{1}{2}\ln\frac{\cosh\lambda - 1}{\cosh\lambda - \cos\beta} - \ln(1-e^{-\lambda}) \quad \text{for} \quad \lambda > 0. \tag{32}$$

Employing (32) in (30) and (31), we obtain

$$\tilde{\Phi} = \frac{S}{4\pi R}\left\{\ln\left[\frac{(\cosh\lambda_1+\cos\beta_1)(\cosh\lambda_2+\cos\beta_1)}{(\cosh\lambda_1-\cos\beta_1)(\cosh\lambda_2-\cos\beta_1)}\right]\right.$$
$$\left. - \ln\left[\frac{(\cosh\lambda_1+\cos\beta_2)(\cosh\lambda_2+\cos\beta_2)}{(\cosh\lambda_1-\cos\beta_2)(\cosh\lambda_2-\cos\beta_2)}\right]\right\} \quad \text{for} \quad \rho > \rho_0, \tag{33}$$

$$\tilde{\Phi} = \frac{S}{4\pi R}\left\{\ln\left[\frac{(\cosh\lambda_1+\cos\beta_1)(\cosh\lambda_3+\cos\beta_1)}{(\cosh\lambda_1-\cos\beta_1)(\cosh\lambda_3-\cos\beta_1)}\right]\right.$$
$$\left. - \ln\left[\frac{(\cosh\lambda_1+\cos\beta_2)(\cosh\lambda_3+\cos\beta_2)}{(\cosh\lambda_1-\cos\beta_2)(\cosh\lambda_3-\cos\beta_2)}\right]\right\} \quad \text{for} \quad \rho < \rho_0. \tag{34}$$

Comparing the two expressions (33) and (34), and noting that $\lambda_2 = -\lambda_3$ when ρ tends to ρ_0, it is obvious that these relations are mathematically identical. Therefore, with the exception of the points of application of the source such as $\rho = \rho_0$, $\theta = \theta_0$ equivalent to $\lambda_2 = \lambda_3 = 0$, and $\beta_2 = 0$, the limits for the function $\tilde{\Phi}$ and its partial derivatives of any order exist as ρ tends to ρ_0.

NUMERICAL RESULTS

In the following, we shall give some numerical results for the case of a beam with four cavities symmetrically located with respect to the ξ and η axes. For all the numerical results presented here, we choose the value of Poisson's ratio ν as 0.3. The function Φ given in relation (4) automatically satisfies the outer boundary condition, and it is an odd function with respect to ξ and even with respect to η. Therefore it is only necessary to satisfy the inner boundary condition at the surface of one of four cavities. This is accomplished as follows. Retaining P basis functions in the series solution (4) and satisfying the boundary condition at $M > P$ points on one of the four inner boundaries, a system of M by P linear algebraic equations is obtained. The system is normalized and solved approximately by the

Table 1. Comparison of the prescribed inner boundary condition (3) with those which take place with the solution for various $\beta_0 = K\pi/18$, and for the case of $a_0 = 0.5$, $\bar{\theta} = \pi/4$ and $\bar{\rho} = 0.1$

K	Prescribed values	Values which are obtained with the solution
1	0.376931	0.376927
3	0.249234	0.249240
5	0.100055	0.100053
7	−0.0329253	−0.0329302
9	−0.127994	−0.127986
11	−0.182165	−0.182168
13	−0.204280	−0.204285
15	−0.205761	−0.205753
17	−0.195104	−0.195109
19	−0.176292	−0.176294
21	−0.147850	−0.147843
23	−0.102319	−0.102323
25	−0.0289317	−0.0289329
27	0.0784839	0.0784889
29	0.211212	0.211209
31	0.341290	0.341287
33	0.429620	0.429625
35	0.444112	0.444110

employment of the method of least square error given by Hildebrand (1956) leading to the determination of constants $A_0, A_1, \ldots, B_1, B_2, \ldots$. For all of the numerical results given here $P = 12$ and $M = 18$. Since the outer boundary condition is automatically satisfied, the satisfaction of the inner boundary condition is a measure of the accuracy of the solution. In Table 1 the satisfaction of the inner boundary condition is shown for the case where $a_0 = 0.5$, $\bar{\theta} = \pi/4$ and $\bar{\rho} = \bar{r}/R = 0.1$.

We shall define the stress concentration S_c as:

$$S_c = \frac{\tau_{\zeta\xi}}{(W/A)}, \tag{35}$$

in which W is the load in the ξ direction, A is the area of the hollow cross-section, and ζ is the axis perpendicular to the $\xi\eta$ plane with $\zeta = 0$ at the fixed end of the cantilever. The

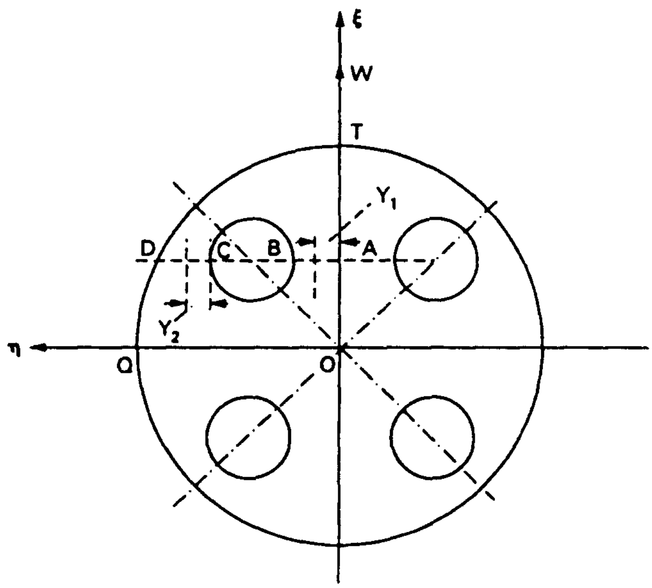

Fig. 2. Positions of lines AB, CD, OT along which the values of S_c are given.

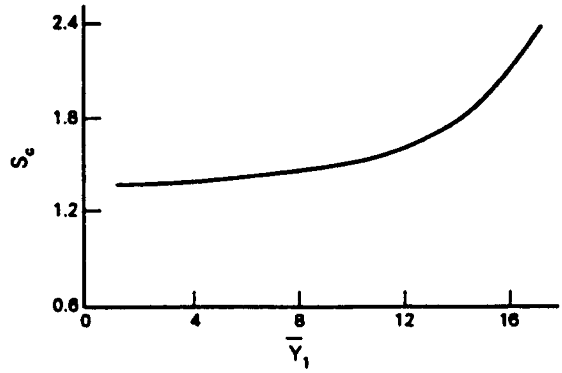

Fig. 3. The values of the stress concentration S_c along AB for $a_0 = 0.5$, $\bar{\theta} = \pi/4$ and $\bar{\rho} = 0.1$.

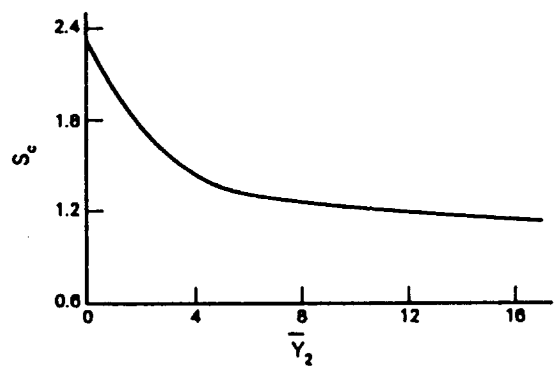

Fig. 4. The values of the stress concentration S_c along CD for $a_0 = 0.5$, $\bar{\theta} = \pi/4$ and $\bar{\rho} = 0.1$.

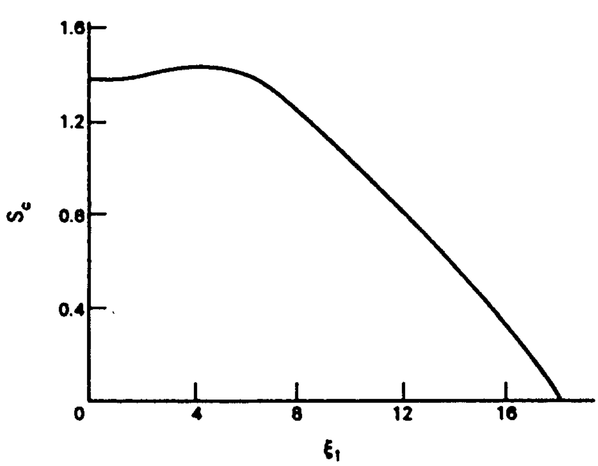

Fig. 5. The values of the stress concentration S_c along OT for $a_0 = 0.5$, $\bar{\theta} = \pi/4$ and $\bar{\rho} = 0.1$.

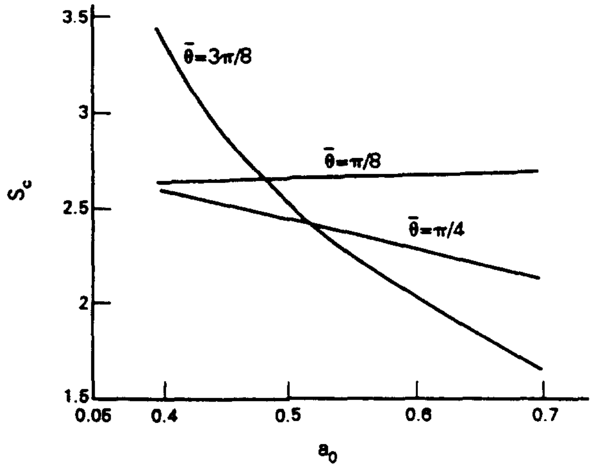

Fig. 6. The values of the stress concentration S_c versus a_0 for $\bar{\rho} = 0.12$ and three different values of $\bar{\theta} = \pi/8$, $\bar{\theta} = \pi/4$ and $\bar{\theta} = 3\pi/8$.

shear stress $\tau_{\zeta\zeta}$ in relation (35) is obtained from the following expression, given in Sokolnikoff (1956):

$$\tau_{\zeta\zeta} = -\frac{W}{2(1+\nu)I_Y}\left[\frac{\partial\Phi}{\partial X} + \tfrac{1}{2}\nu X^2 + (1-\tfrac{1}{2}\nu)Y^2\right], \tag{36}$$

in which I_Y is the area moment of inertia with respect to the Y axis.

In Figs 3 and 4 the values of the stress concentration S_c along AB and CD (see Fig. 2) are plotted versus the dimensionless distances $\bar{Y}_1 = 17Y_1/AB$, and $\bar{Y}_2 = 17Y_2/CD$, respectively. For both cases $a_0 = 0.5$, $\bar{\theta} = \pi/4$ and $\bar{\rho} = 0.1$. In Fig. 5 the values of the stress concentration S_c along OT are graphed versus $\xi_1 = 18\xi$ for the case where $a_0 = 0.5$, $\bar{\theta} = \pi/4$ and $\bar{\rho} = 0.1$. These numerical results indicate that S_c is maximum at point B. This maximum value grows when the boundaries of the two circular cut-outs with angles $\bar{\theta}$ and $\pi - \bar{\theta}$ come close to each other and the ξ axis. In Fig. 6 the values of stress concentration S_c versus a_0 are plotted for the cases of $\bar{\rho} = 0.12$ and three different values of $\bar{\theta} = \pi/8$, $\bar{\theta} = \pi/4$ and $\bar{\theta} = 3\pi/8$.

CONCLUSION

The numerical results obtained in this investigation are very accurate and well within the usual engineering approximations. It is seen from Table 1 that the relative error in satisfaction of the inner boundary condition is of the order of 10^{-5}. However, for bigger values of $\bar{\rho}$ and when the boundaries of the cavities are closer to the ξ axis, this relative error is somewhat higher. For example, for the case where $a_0 = 0.5$, $\bar{\theta} = 3\pi/8$, $\bar{\rho} = 0.12$ the relative error is of the order of 10^{-2}. The aforementioned error discussion is the only measure in judging the correctness and accuracy of the solution presented in this investigation. This is due to the fact that there are no other published results for the bending of a cantilever with the geometry considered here. The closed-form basis functions lead to the consumption of much less computer time and to much more accurate results.

Finally, it is believed that the problem of a semicircular beam with cavities may be solved with similar basis functions.

REFERENCES

Dwight, H. B. (1957). *Table of Integrals and other Mathematical Data*. Macmillan, New York.
Gradshteyn, I. S. and Ryzhik, I. M. (1965). *Table of Integral Series and Products*. New York and London.
Hildebrand, F. B. (1956). *Introduction to Numerical Analysis*. McGraw-Hill, New York.
Love, A. E. H. (1906). *Mathematical Theory of Elasticity*, 2nd edn.
Naghdi, A. K. (1973). The effect of a transverse shear acting on the edge of a circular cutout in a simply supported circular cylindrical shell. *Ing.-Arch.* 42.

Naghdi, A. K. (1988). Certain basis functions for biharmonic and Laplace's equations and applications. *Lecture Notes in Engineering*, Vol. 39. Springer, New York.
Seth (1936). *Proc. Ind. Acad. Sci.* 4.
Sokolnikoff, S. (1956). *Mathematical Theory of Elasticity*. McGraw-Hill, New York.
Stevenson, A. C. (1949). The center of flexure of a hollow shaft. *Proc. London Mathematical Soc.*
Timoshenko, S. and Goodier, J. N. (1951). *Theory of Elasticity*. McGraw-Hill, New York.
Young, Elderton and Pearson (1918). *Draper's Co. Research Memoirs*, Tech. Series, No. VII.